An Interpretation of Aristotelian Logic According to George Boole

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Abstract: *This article is placed in the general frame of philosophy of information and it attempts to present a rarely mentioned application of Boolean system to the reduction of logic to mathematics in opposition to the widely known initiative of reduction of mathematics to logic (Frege, Russell, Wittgenstein, Godel). It concerns the transformation of categorical propositions to equivalent algebraic equations aiming at the extraction of some types of valid logical structures, including inversion formulae and syllogism schemata, through elimination and reduction techniques, presented in 'Laws of Thought' of George Boole.*

Keywords: *Aristotelian logic, Boolean system, Syllogism*.

I. INTRODUCTION

In 19th century, mathematicians and logicians were intensely interested in the discovery of some fundamental principles, which could be used as a basis for strict extraction and proof of all the true mathematical and logical propositions. To this purpose, two approaches appeared, reduction of mathematics to logic and reduction of logic to mathematics. Frege was a prominent philosopher and mathematician who tried in all his life to ground arithmetic to logic by connecting natural numbers to senses and defining a number of a sense F as the extension of the sense 'of equal multitude to sense F' or the set of all the sets which have elements with 1-1 correspondence to the elements of F. Thus an arithmetic object is expressed through 1-1 correspondence, which is a logical process.

In the same period, George Boole, another philosopher and mathematician, was moving to the opposite direction, trying to reduce logical processes into arithmetic operations. Categorical propositions are expressed in the form of algebraic equations, whose variables represent objects of the extension of senses. Development, elimination and reduction techniques are used aiming at the extraction of an equation out of a number of given equations. The proposition corresponding to the extracted equation is the logical result of the given propositions. This methodology is applied for the presentation of Aristotelian inversion rules and the extraction of the valid syllogistic moods.

In Chapter 2 a short presentation of the ancient Aristotelian and the medieval scholastic syllogistic figures is given.

In Chapter 3 the basic formalism of Boolean system and the conclusion mechanism is given.

In Chapter 4 the Boolean methodology is applied aiming at the proof of the basic inversion formulae and rules for the synthesis of the valid syllogistic forms. The classical Aristotelian syllogism in all variations of valid figures and forms can be seen as one of multiple available implications of the premises. Next, the 19 valid syllogistic forms are produced through the general rules.

Lastly, in Chapter 5 some extensions of Aristotelian syllogism and alternative interpretations of the logical-algebraic equations of syllogism are presented.

II. THE ARISTOTELIAN SYLLOGISTIC FIGURES AND THE EXTENDED SCHOLASTIC SYSTEM

According to Aristotle's Prior Analytics, "syllogism is discourse in which, certain things being stated, something other than what is stated follows of necessity from their being so. I mean by the last phrase that they produce the consequence, and by this, that no further term is required from without in order to make the consequence necessary" (Συλλογισμός δέ εστι λόγος εν ώ τεθέντων τινών έτερόν τι των κειμένων εξ ανάγκης συμβαίνει τώ ταύτα είναι)[87].

Syllogism consists of three categorical propositions, of which two are given (major premise and minor premise) and the third one is inferred from them (inference or conclusion). The stated things are expressed in the premises and the following one in the conclusion. The major premise (1st proposition) includes the predicate term and the middle term, the minor premise (2nd proposition) includes the subject term and the same middle term and the conclusion (3rd proposition) includes the subject term and the predicate term (extreme terms).

Aristotle categorises the categorical propositions from quality view into affirmative and negative and from the quantity view into universal and particular. This double categorisation leads to the following four possible types of categorical propositions: AaB (all A are B), AeB (no A are B), AiB (some A are B) και AoB (some A are not B), following the representation of medieval scholastics.

Using the scholastic copula symbolism, the four figures (including also the fourth not explicitly mentioned by Aristotle) are the following:

1st figure : MP + SM → SP
2nd figure : MP + MS → SP
3rd figure : PM + SM → SP

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[87] Aristotle, Prior Analytics, part 1.

4^{th} figure : PM + MS → SP

The terminology of the 19 valid forms (moods) of the 4 figures according to the medieval tradition (Pantzig, 1968), (Παπανούτσος, 1974) are:

1^{st} figure

Barbara (γράμματα), Celarent (έγραψε), Darii (γραφίδι), Ferio (τεχνικός)

2^{nd} figure

Cesare (έγραψε), Camestres (κάτεχε), Festino (μέτριον), Baroco (άχολον)

3^{rd} figure

Darapti (άπασι), Felapton (σθεναρός), Disamis (ισάκις), Datisi (ασπίδι), Bocardo (ομαλός), Ferison (φέριστος)

4^{th} figure

Bamalip (άπασι), Calemes (πάρεχε), Dimatis (ισάκις), Fesapo (έπαθλον), Fresison (σέλινον)

The vowels of the words imply the type of the categorical proposition (a,e,i,o).

III. THE BOOLEAN LOGICAL ALGEBRAIC SYSTEM

A. Propositions, variable, operations

The system to be described was developed by George Boole in his important work 'On the laws of thought' (Boole, 1854).

It concerns propositions of the form 'x is y' (primary or predicative propositions) and 'x then y' (secondary or conditional propositions). In primary propositions the related items are terms, while in the secondary propositions the related items are propositions. In primary propositions the variables x, y, etc. represent categories of things or things of a certain property or extensions of senses, for instance people, animals, white things. In secondary proposistions they represent confirmable through experience facts, for instance 'the sun emits light' or philosophical suggestions, for instance 'God acts freely'.

In the present work only primary propositions will be studied. It should be mentioned that in the frame of the Boolean system all propositions are reduced to the 'x is y' form. Thus the basic logical relation 'subject-predicate' is reserved, but it is managed in strict mathematical way. Boole denotes the border between traditional logic and contemporary formal logic, since he is the first who tried the application of algebraic formalism to the frame of logic. His system was not paid much attention, as Frege introduced the sense of function, where the copula relation becomes one specific case between various possible functions. Later, predicative calculus of Frege – Russell prevailed. On the other hand, Boolean system, as it is structured in the form 'subject-predicate' (relation of the extensions of the relative senses) is quite appropriate for a scientific analysis of the Aristotelian propositions and the syllogistic schemes.

The product of variables xy represents a composite category of things, which possess both properties (intersection of categories). The product xy of the white things x and the valuable things y represents the category of white valuable things. The transposition property xy=yx is obviously valid in Boolean system since the first form represents the white valuable things, while the second one represents the valuable white things, that is the same category.

The product xx or x^2 represents the things that possess property x and property x, a category which is the same with the things that possess property x. Therefore the equation $x^2 = x$ is an identity (valid for all x). An interesting observation is that the integer solutions of this identity are only 0 and 1, the only constants of the Boolean systems with an important logical interpretation. 0 represents the empty category, that is the category with no element and 1 represents the universal category, that is the category including all the things.

The sum of variables x+y represents a composite category of things, which possess one or the other property or both of them (union of categories). The sum x+y of the male people x and the female people y represents the category of all the people. The transposition property x+y=y+x is obviously valid in Boolean system since the first form represents the category including men and women and the second form represents the category including women and men.

The subtraction of variables x-y represents the residual category, which results with the withdrawal of the things with property y from the wider category of things with poperty x. A special case is the category 1-x, which contains all the things, which do not possess property x. The identity $x^2 = x$ is equivalent to x(1-x)=0, which is an algebraic representation of the basic Aristotelian principle of contradiction.

B. Proposition types

The four types of Aristotelian propositions are transformed as follows:

y a x (all y are x)

$$y = ux \tag{1}$$

y e x (no y are x)

$$y = u(1-x) \tag{2}$$

y i x (some y are x)

$$vy = ux \tag{3}$$

y o x (some y are not x)

$$vy = u(1-x) \tag{4}$$

In Eq. (1) – (4) variables u and v mean *some*, since the extensions of categories in all the categorical propositions do not coincide and the relation is inclusion.

The three basic tools of Boolean system that will be used are:

- development of a function by means of one or more variables

- elimination of a variable from an equation / proposition
- reduction of an equation / proposition from a set of equations / propositions

IV. EXTRACTION OF BASIC LOGICAL STUCTURES THROUGH BOOLEAN SYSTEM

The basic inversion formulae of Aristotelian logic are produced through the elimination of ‘some’ variable, solution in terms of the interested variable and the development of the corresponding function (Boole, 1854).

The inversion formulae to be proved are : AaB $\rightarrow$ BiA, AeB $\rightarrow$ BeA and AiB $\rightarrow$ BiA.

AaB $\rightarrow$ A=uB $\rightarrow$ A(1-B)=0 $\rightarrow$ B =A/A $\rightarrow$ B = A + 0/0 (1-A), that is the category of B includes all the category of A and an indefinite part of the category of non A, or some B are A $\rightarrow$ BiA.

AeB $\rightarrow$ A = u(1-B) $\rightarrow$ AB=0 $\rightarrow$ B = 0/A $\rightarrow$ B = 0/0 (1-A) $\rightarrow$ B = u(1-A) $\rightarrow$ BeA.

AiB $\rightarrow$ vA=uB $\rightarrow$ uB=vA $\rightarrow$ BiA.

The basic syllogistic rules of Aristotelian logic are produced through the reduction of a proposition from the set of the two premises, the produced presence of the three categories in one equation and then the elimination of the middle term, which results to a relation between the extreme terms (Boole, 1854).

Boole, in order to produce the syllogistic rules, expresses the premises in a general form, which includes as special cases all the four possible types of categorical propositions Eq. (1) – (4).

A. Case 1

The general equations of syllogism, where the middle term has the same quality in both premises are:

$$u x = u' y \quad (5)$$

$$w z = w' y \quad (6)$$

where u, u', w και w' are either 1 or ‘some’ variables, x, z are the extreme terms and y is the middle term.

It is not allowed in each of Eq. (5), (6) the simultaneous absense of both the ‘some’ variables. This would lead to definition type propositions and not to categorical propositions (the only allowed type in syllogism). Besides, concerning quantity, x,y,z have to be considered as categories that do or do not possess a property, and therefore 1-x,1-y,1-z mean the contrary categories.

The result is a logical relation between extreme terms x and z. To this purpose, we extract from Eq. (5) and (6) a new equation, and then we eliminate y. After this, we solve for x, 1-x and ux and we develop the produced function in terms of z. This leads to an expression, which contains the contrary categories z and 1-z. The coefficients of z and 1-z contain u, u´, w, w´. Therefore the participation or not of the categories z and 1-z, as well as the extension of participation depends on values and relations of u, u´, w, w´.

After the reduction, elimination and development processes, the produced equations for x, 1-x, ux are:

$$x = [uu'ww'+0/0[uu'(1-w)(1-w')+ww'(1-u)(1-u')+(1-u)(1-w)]]\, z + 0/0\, [uu'(1-w')+1-u]\, (1-z) \quad (7)$$

$$1-x = [u(1-u')[ww'+(1-w)(1-w')]+u(1-w)w'+0/0[uu'(1-w)(1-w')+ww'(1-u)(1-u')+(1-u)(1-w)]]z+ [u(1-w)w'+ 0/0[uu'(1-w')+1-u]]\, (1-z) \quad (8)$$

$$ux = [uu'ww'+0/0.uu'(1-w)(1-w')]\, z + 0/0.(1-w')\, (1-z) \quad (9)$$

In the second member of the conclusion in Eq. (7)-(9) both categories, z and 1-z, are present. This is not a valid form of a syllogism conclusion, where the one extreme term x is related to the other extreme term in its affirmative (z) or negative (1-z) form, according to the accepted categorical propositions Eq. (1)-(4).

Equation (9) is a valid conclusion if w’ = 1. It means that the middle term y participates in one of the premises universally. Then Eq. (5), (6), (9) become as follows:

$$u x = u' y \quad (10)$$

$$y = w z \quad (11)$$

$$ux = uu'w z \quad (12)$$

From Eq. (11), it is implied that w≠1, since otherwise it would not be a categorical proposition. Thus, in Eq. (12) uu’w≠1. Therefore, extreme term z participates in both premise and conclusion particularly.

The above are summated in the following first syllogistic rule:

“In the case that the middle term has the same quality in both premises and participates in at least one of the premises universally, then the extreme terms maintain in the conclusion the same quality and quantity as in the premises”.

B. Case 2

The general equations of syllogism, where the middle term has opposite qualities in the premises are:

$$u x = u' y \quad (13)$$

$$w z = w' (1-y) \quad (14)$$

After the reduction, elimination and development processes, the produced equations for x, 1-x, ux are:

$$x = [uu'(1-w)w'+0/0[ww'(1-u)+(1-u)(1-u')(1-w)+u'(1-w)(1-w')]]\, z + [uu'w'+0/0[(1-u)(1-u')+u'(1-w')]]\, (1-z) \quad (15)$$

$$1-x = [ww'u+u(1-u')(1-w)+0/0[ww'(1-u)+(1-u)(1-u')(1-w)+u'(1-w)(1-w')]]\, z + [u(1-u')+0/0[u'(1-w')+(1-u)(1-u')]]\, (1-z) \quad (16)$$

$ux=[uu'(1-w)w'+0/0uu'(1-w)(1-w')]z+[uu'w'+0/0.uu'(1-w')](1-z)$ (17)

Equation (17) is a valid conclusion if w = 1. It means that the extreme term z participates in one of the premises universally. Then Eq. (13), (14), (17) become as follows:

$$ux = u'y \quad (18)$$

$$z = w'(1-y) \quad (19)$$

$$ux = [uu'w'+0/0.uu'(1-w')](1-z) \quad (20)$$

From Eq. (19), it is implied that $w'\neq 1$, since otherwise it would not be a categorical proposition. Besides, from Eq. (18), it is implied that u=1 and u'=1 cannot hold simultaneously, since otherwise it would not be a categorical proposition. Thus, in Eq. (20) $uu'w\neq 1$. Therefore, the extreme term z participates in the premise Eq. (19) in affirmative universal way and in the conclusion Eq. (20) in negative particular way.

The above are summated in the following second syllogistic rule:

"In the case that the middle term has opposite qualities in the premises and at least one of the extreme terms participates in one premise universally, then this extreme term participates with the opposite quality and quantity in the conclusion, while the other extreme term maintains the same quality and quantity as in the premises".

Equation (16) is a valid conclusion if u=1 and w' = 1. It means that the middle term participates in both premises universally. Then (13), (14), (16) become as follows:

$$y = ux \quad (21)$$

$$1-y = wz \quad (22)$$

$$1-x = [uw+0/0(1-u)w]\, z \quad (23)$$

From Eq. (21), (22) it is implied that $u\neq 1$ and $w\neq 1$, since otherwise they would not be categorical propositions. Thus, in Eq. (23) $uw\neq 1$. Therefore, the one extreme term x participates in the conclusion in negative universal way, that is with opposite quality and quantity, while the other extreme term z in affirmative particular way, that is with the same quality and quantity as in the premises.

The above are summated in the the following third syllogistic rule:

"In the case that the middle term has opposite qualities in the premises and it participates universally in both of them, then one of the extreme terms participates with the opposite quality and quantity in the conclusion, while the other extreme term maintains the same quality and quantity as in the premises".

The 4 valid moods of 1st syllogistic figure are proved via the 1st syllogistic rule. The 4 valid moods of 2nd syllogistic figure are proved via the 2nd syllogistic rule. The 6 valid moods of 3rd syllogistic figure are proved via the 1st syllogistic rule. The 5 valid moods of 4th syllogistic figure are proved proved via the 1st and 2nd syllogistic rules.

V. EXTENSIONS OF ARISTOTELIAN SYLLOGISM

Extending the traditional Aristotelian syllogism, Boole (1854) also studied propositions, where the subject represents not a category but a complementary category, for instance not animals. Accepting these propositions, additional moods of syllogism are produced:

y=ux (all y's are x's) (24)

1-y=wz (all non y are z's) (25)

→

1-x=vz (all non x are z') (26)

1-z=v'x (all non z are x's) (27)

due to the third rule

Finally, Boole (1854) suggests an alternative interpretation of the general equations (5), (6) quite unrelated to the syllogism frame. Instead of using u, u', w, w' as *whole* (value=1) or *some*, we might consider that they represent categories of things, like x, y, z. In this case the produced equations, like Eq. (7), would correspond to a logical proposition declaring the extension relation of the 7 categories x, y, z, u, u', v, v'.

VI. CONCLUSION

In the historical contest between the philosophers who considered logic or mathematics as the ultimate grounding level of scientific truth, overwhelmed those that reduced mathematics to logic (Frege, Whitehead, Russell, Wittgenstein, Boolos). George Boole was pioneer in expressing logic in terms of arithmetic, but his representation rules and operations did not attract extensive attention due to some consistency limitations exhibited by Jevons (the problem of x+x=x). Boolean concept for representation of logical terms through mathematical variables found prolific applications in the case of formal binary logic (de Morgan, Schroder, Pierce, Shannon).

In this paper it was attempted to present an application of the algebraic techniques, introduced by George Boole in his monumental work "The Laws of Thought", to the mathematical proof of Aristotelian logical schemata. This methodology appeared to be very fruitful, as it led to a very delicate demonstration of the basic logical structures. My suggestion is that Boolean system could be used for an alternative compact way of presentation and instruction of basic structures of Logic.

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